Ex. Find $f$ if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, $f(1) = 1$

Find first derivative (antiderivative of $f''$).

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C.$$

Find $f$.

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D$$

$$\begin{array}{c} f \\ \text{antider} \uparrow \quad \downarrow \text{der} \\ f' \\ \text{antider} \uparrow \quad \downarrow \text{der} \\ f'' \end{array}$$

Since $\begin{cases} f(0) = 4 \\ f(1) = 1 \end{cases} \Rightarrow \begin{cases} 0 + 0 - 0 + 0 + D = 4 \\ 1 + 1 - 2 + C.1 + D = 1. \end{cases}$

$$\Rightarrow \begin{cases} D = 4 \\ C = -3 \end{cases}$$

$$\Rightarrow f(x) = x^4 + x^3 - 2x^2 - 3x = 4.$$

Linear motion

$s(t)$ : position function $\qquad s'(t) = v(t)$

$v(t)$ : velocity function $\qquad v'(t) = a(t)$

$a(t)$ : acceleration function

**EXAMPLE 6** A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

$a(t) = 6t + 4 \Rightarrow v(t) = 6\frac{t^2}{2} + 4t + C. = 3t^2 + 4t + C$

Since $v(0) = -6 \Rightarrow 3.0 + 4.0 + C = -6.$

$\Rightarrow C = -6.$

$\Rightarrow v(t) = 3t^2 + 4t - 6.$

$\Rightarrow s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D$

Since $s(0) = 9 \Rightarrow 0 + 0 - 0 + D = 9$

$\Rightarrow s(t) = t^3 + 2t^2 - 6t + 9.$

**EXAMPLE 7** A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff, 432 ft above the ground. Find its height above the ground $t$ seconds later. When does it reach its maximum height? When does it hit the ground?

**EXAMPLE 7** A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff, 432 ft above the ground. Find its height above the ground $t$ seconds later. When does it reach its maximum height? When does it hit the ground?

Gratitational force produce a downward accelaration denoted by $g$

$$g = 9.8\, m/s^2 \ (\text{or } 32\, ft/s^2)$$

We have $a(t) = v'(t) = -32.$

$\Rightarrow \quad v(t) = -32t + C.$

Since $v(0) = 48.$, we have $v(0) = -32.0 + C = 48 \Rightarrow C = 48.$

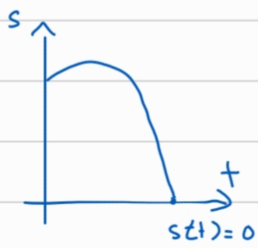
Thus $v(t) = -32t + 48$

The maximum height is reached when $v(t) = 0$, that is

$$-32t + 48 = 0 \Leftrightarrow 32t = 48 \Leftrightarrow t = \frac{48}{32} = 1.5 \text{ s}$$

19/24

Since $s'(t) = v(t) \Rightarrow s(t) = 32\frac{t^2}{2} + 48t + D$




Since $s(0) = 432\, ft$ we have

$16.0 + 48.0 + D = 432 \Rightarrow D = 432.$

Thus $s(t) = -16t^2 + 48t + 432.$

The ball hit the ground when $s(t) = 0.$

$\Rightarrow \quad -16t^2 + 48t + 432.$

$\Leftrightarrow \quad t^2 - 3t - 27 = 0$

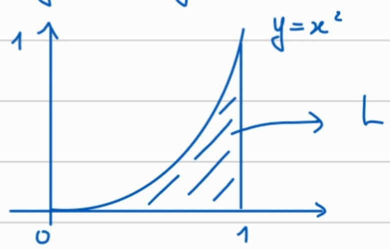
$\Rightarrow \quad t = \dfrac{3 \pm 3\sqrt{13}}{2}$

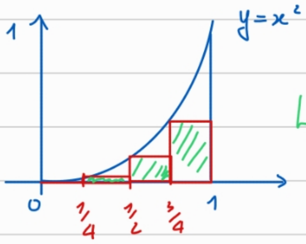
$t > 0 \Rightarrow \quad t = \dfrac{3 + 3\sqrt{13}}{2} \approx 6.9$ s.

---

## § 5. The area problem.

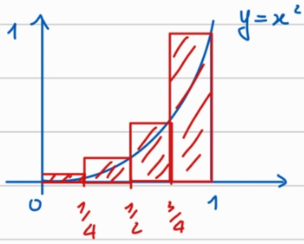
Ex 1. Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.




$L_4$

$R_4$.




Divide $[0,1]$ into 4 subintervals. $[0, 1/4], [1/4, 1/2], [\frac{1}{2}, \frac{3}{4}], [3/4, 1]$.

Use left end points to estimate

$$L_4 = \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2$$

$$= \frac{7}{32} \approx 0.21$$

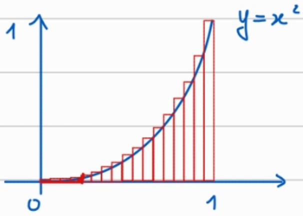
Use right endpoints.

$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot 1^2$$

$$= \frac{15}{32} \approx 0.46.$$

$$\Rightarrow 0.21 \approx L_4 \leq L \leq R_4 \approx 0.46$$

Using more subintervals, taking the limit, we have the area. L.




Divide $[0,1]$ by $n$ intervals.

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$

$$R_n = \frac{1}{n} f\left(\frac{1}{n}\right) + \dots + \frac{1}{n} f\left(\frac{n}{n}\right)$$

$$\Rightarrow R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \dots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \dots + n^2\right)$$



$$= \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6n^2}$$

$$\text{Area } L = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2}$$

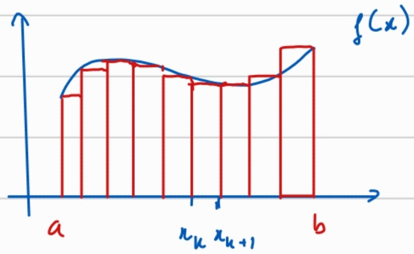
$$= \lim_{n\to\infty}\frac{\cancel{n^2}\left(2 + 3/n + 1/n^2\right)}{\cancel{n^2}\cdot 6}$$

$$= \lim_{n\to\infty}\frac{2 + 3/n + 1/n^2}{6}$$

$$= \frac{1}{3}.$$

$$\Rightarrow \text{Area} = \frac{1}{3}.$$

## General.




Divide $[a,b]$ into $n$ subintervals

$$\Delta x = \frac{b-a}{n}.$$

endpoints of intervals: $[x_k, x_{k+1}]$

The right endpoints:

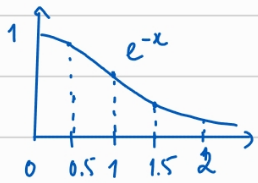
$$x_1 = a + \Delta x$$
$$x_2 = a + 2\Delta x$$
$$\vdots$$
$$x_k = a + k\Delta x$$
$$\vdots$$

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x.$$

**Def**: The area of the region $S$ that lies under the graph of a continuous function $f$ is the limit of the sum of the areas of approximating rectangles

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[ f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x\right]$$

Ex: Estimate the area under graph of $e^{-x}$ by 4 subintervals using mid points. $x \in [0, 2]$.




$n = 4$. $\Delta x = \frac{2-0}{4} = 0.5$.

We have 4 subintervals:

$[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, $[1, \frac{3}{2}]$, $[\frac{3}{2}, 2]$

Midpoints: $\frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$, $\frac{7}{4}$.

$$M_4 = \sum_{i=1}^{4} f(x_i^*)\Delta x = f(\tfrac{1}{4})\cdot 0.5 + f(\tfrac{3}{4})\cdot 0.5 + f(\tfrac{5}{4})\cdot 0.5 + f(\tfrac{7}{4})\,0.5$$

$$= 0.5\left(e^{-(\frac{1}{4})} + e^{-(\frac{3}{4})} + e^{-\frac{5}{4}} + e^{-\frac{7}{4}}\right)$$



## § 5.2. The definite integral.

## § 5.2. The definite integral.

**2 Definition of a Definite Integral** If $f$ is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into $n$ subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 (= a), x_1, x_2, \ldots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so $x_i^*$ lies in the $i$th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of $f$ from $a$ to $b$** is

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that $f$ is **integrable** on $[a, b]$.

$$\int_a^b f(x)\, dx = \int_a^b f(t)\, dt = \int_a^b f(r)\, dr.$$

**4 Theorem** If $f$ is integrable on $[a, b]$, then

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\, \Delta x$$

where $\quad \Delta x = \dfrac{b-a}{n} \quad$ and $\quad x_i = a + i\,\Delta x$

### Evaluating definite integrals

**Sums of Powers**

5 $\displaystyle\sum_{i=1}^{n} 1 = n$

6 $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

7 $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

8 $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

**Properties of Sums**

9 $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$

10 $\displaystyle\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

11 $\displaystyle\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Ex. Evaluate $\displaystyle\int_0^3 (x^3 - 6x)\, dx$.

$f(x) = x^3 - 6x, \quad a = 0, \ b = 3.$

$\Delta x = \dfrac{b-a}{n} = \dfrac{3}{n}.$

The endpoints are: $x_0 = 0, \ x_1 = \dfrac{3}{n}, \ x_i = \dfrac{3i}{n}.$

Thus: $\displaystyle\int_0^3 (x^3 - 6x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$

Thus : $\displaystyle\int_0^3 (x^3-6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 6\,\frac{3i}{n}\right]$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\left[\frac{27}{n^3}i^3 - \frac{18\,i}{n}\right]$$

$$= \lim_{n\to\infty} \frac{81}{n^4}\sum_{i=1}^{n} i^3 - \lim_{n\to\infty}\frac{54}{n^2}\sum_{i=1}^{n} i$$



$$\Rightarrow \int_0^3 (x^3-6x)\,dx = \lim_{n\to\infty}\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \lim_{n\to\infty}\frac{54}{n^2}\,\frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty}\frac{81}{4}\left[\frac{n^2+n}{n^2}\right]^2 - \lim_{n\to\infty}\frac{54}{2}\,\frac{n^2+n}{n^2}$$

$$= \lim_{n\to\infty}\frac{84}{4}\left[\frac{\cancel{n^2}(1+1/n)}{\cancel{n^2}}\right]^2 - \lim_{n\to\infty}\frac{54}{2}\,\frac{\cancel{n^2}(1+1/n)}{\cancel{n^2}}$$

$$= \frac{84}{4} - \frac{54}{2} = -\frac{27}{4} = -6.75.$$

$$\Rightarrow \int_0^3 (x^3-6x)\,dx = -6.75.$$

## Properties of definite integral.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

$$\int_a^a f(x)\,dx = 0.$$

**Properties of the Integral**

1. $\int_a^b c\,dx = c(b-a)$, where $c$ is any constant

2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where $c$ is any constant

4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

**5.** $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

**Comparison Properties of the Integral**

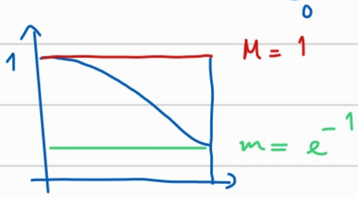
**6.** If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

**7.** If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

**8.** If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$$

Ex. Estimate $\int_0^1 e^{-x^2}\,dx$ by property 8.




We have $m = e^{-1}$, $M = 1$

$m = e^{-1} \leq f(x) \leq M = 1.$

$\Rightarrow \quad e^{-1}(1-0) \leq \int_0^1 f(x)\,dx \leq 1.(1-0).$

$\Rightarrow \quad e^{-1} \leq \int_0^1 f(x)\,dx \leq 1.$